Review of Probability
Wilks, Chapter 2

AOSC630
Spring, 2008
Definition

- **Event**: set, class, or group of possible uncertain outcomes
  - *A compound event* can be decomposed into two or more (sub) events
  - *An elementary event* cannot

- **Sample space (event space), S**
  - The set of all possible elementary events

- **MECE (Mutually Exclusive & Collectively Exhaustive)**
  - *Mutually Exclusive*: no more than one of the events can occur
  - *Collectively Exhaustive*: at least one of the events will occur

  ➔ *A set of MECE events completely fills a sample space*
Probability Axioms

- $P(A) \geq 0$
- $P(S) = 1$
- If $(E_1 \cap E_2) = 0$, i.e., if $E_1$ and $E_2$ exclusive,
  then $P(E_1 \cup E_2) = P(E_1) + P(E_2)$
Probability

- Probability ~ Frequency
  - $P(E) = \lim_{n \to \infty} \frac{\#E = \text{yes}}{\text{total} \_ n}$

- If $E_2 \subseteq E_1$, then $P(E_1) \geq P(E_2)$

- $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

Venn Diagrams

$E_1 \quad E_2 \quad E_1 \cup E_2$
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**Recall Threat Score (TS)**

$$\frac{P(F = \text{yes} \cap \text{Ob} = \text{yes})}{P(F = \text{yes} \cup \text{Ob} = \text{yes})}$$

**Venn Diagrams**
Conditional Probability

- **Probability of $E_1$ given that $E_2$ has happened**

\[
P(E_1|E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)}
\]

\[
P(E_1 \cap E_2) = P(E_1|E_2)P(E_2) \quad \text{; Multiplicative Law}
\]

- **Independent Event**
  - The occurrence or nonoccurrence of one does not affect the probability of the other

\[
P(E_1 \cap E_2) = P(E_1)P(E_2)
\]

i.e. \[
P(E_1|E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)} = P(E_1)
\]
Exercise

- From the Penn State station data for Jan. 1980, compute the probability of precipitation, of $T>32F$, conditional probability of precipitation if $T>32F$, and conditional probability of precipitation tomorrow if it is raining today.

- Prove graphically the DeMorgan Laws:

$$ P\left((A \cup B)^c\right) = P\left(A^c \cap B^c\right); P\left((A \cap B)^c\right) = P\left(A^c \cup B^c\right) $$
Total Probability

- MECE events, \( \{E_i\}, \ i=1, \ldots, I \)

\[
P(A) = \sum_{i=1}^{I} P(A \cap E_i) = \sum_{i=1}^{I} P(A|E_i)P(E_i)
\]
Bayes’ Theorem

- Bayes’ theorem is used to “invert” conditional probabilities
  - If $P(E_1|E_2)$ is known, Bayes’ Theorem may be used to compute $P(E_2|E_1)$.

\[
P(E_i|A) = \frac{P(E_i \cap A)}{P(A)} = \frac{P(A|E_i)P(E_i)}{P(A)} = \frac{P(A|E_i)P(E_i)}{\sum_{j=1}^{I} P(A|E_i)P(E_j)}
\]

- Combines prior information with new information
Example of Bayesian Reasoning

- Relationship between precipitation over SE US and El Nino
  - Precipitation Events: $E_1$(above), $E_2$(normal), $E_3$(below) are MECE
  - El Nino Event: $A$
  - Prior information (from past statistics)
    - $P(E_1)=P(E_2)=P(E_3)=33\%$
    - $P(A|E_1)=40\%$; $P(A|E_2)=20\%$; $P(A|E_3)=0\%$
Example of Bayesian Reasoning

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![Bayesian Reasoning Diagram]
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\[
P(A|E_1) = \frac{P(A \cap E_1)}{P(E_1)} = \frac{0.33 \times 0.40}{0.33} = 0.40
\]
Example of Bayesian Reasoning

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  - El Nino Event: A
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    - $P(E_1)=P(E_2)=P(E_3)=33\%$
    - $P(A | E_1)=40\%$; $P(A | E_2)=20\%$; $P(A | E_3)=0\%$

\[
P(A | E_1) = \frac{P(A \cap E_1)}{P(E_1)} = \frac{0.33 \times 0.40}{0.33} = 0.40
\]

\[
P(A | E_2) = \frac{P(A \cap E_2)}{P(E_2)} = \frac{0.33 \times 0.20}{0.33} = 0.20
\]
Example of Bayesian Reasoning

- **Total probability of A**

  \[ P(A) = \sum_{i=1}^{3} P(A | E_i)P(E_i) \]
  \[ = P(A | E_1)P(E_1) + P(A | E_2)P(E_2) + P(A | E_3)P(E_3) \]
  \[ = 0.4 \times 0.33 + 0.2 \times 0.33 + 0 \times 0.33 = 0.20 \]

- **NEW information**: El Nino is happening!
  Probability of above normal precipitation?

  \[ P(E_1 | A) = \frac{P(A \cap E_1)}{P(A)} \]

A

E₁ 33%  E₂ 33%  E₃ 33%
Example of Bayesian Reasoning

- **Total probability of A**

\[
P(A) = \sum_{i=1}^{3} P(A | E_i)P(E_i)
\]

\[
= P(A | E_1)P(E_1) + P(A | E_2)P(E_2) + P(A | E_3)P(E_3)
\]

\[
= 0.4 \times 0.33 + 0.2 \times 0.33 + 0 \times 0.33 = 0.20
\]

- **NEW information**: El Nino is happening!

Probability of above normal precipitation?

\[
P(E_1 | A) = \frac{P(A \cap E_1)}{P(A)}
\]

\[
= \frac{0.4 \times 0.33}{0.2} = 0.66
\]
Example of Bayesian Use in Variational Data Assimilation

- **Prior knowledge (measurement or forecast)**
  - $T_1$ of the true value $T$

- **New measurement, $T_2$**

  \[
P(T | T_2) = \frac{P(T_2 | T) P_{\text{prior, given } T_1}(T)}{P(T_2)} = \frac{\frac{1}{\sqrt{2\pi \sigma_1}} e^{-\frac{(t_2 - T)^2}{2\sigma_1^2}} \cdot \frac{1}{\sqrt{2\pi \sigma_2}} e^{-\frac{(t_2 - T)^2}{2\sigma_2^2}}}{\frac{1}{\sqrt{2\pi \sigma_2}} e^{-\frac{(t_2 - T)^2}{2\sigma_2^2}}}
\]

  $\therefore P(T \cap T_2) = P(T_2 | T) P(T)$

  **Note** The total probability of a measurement $T_2$ given a climatological average $\overline{T}$ is independent of $T$

**Our best estimate of the true temperature $\overline{T}$:**
the value that maximizes (over $\overline{T}$) the probability $P(T | T_2)$

\[
\log P(T | T_2) = \text{const} - \frac{(T_2 - T)^2}{2\sigma_2^2} - \frac{(T - T_1)^2}{2\sigma_1^2}
\]

**Or minimizes (over $\overline{T}$) the cost function used in 3D-Var**

\[
J = \frac{(T_2 - T)^2}{2\sigma_2^2} + \frac{(T - T_1)^2}{2\sigma_1^2}
\]
Probability Density Function

- Gaussian distribution with mean, $m_k$, & variance, $\sigma_k$

$$f_k(x) = \frac{1}{\sqrt{2\pi\sigma_k^2}} \exp\left(-\frac{(x-m_k)^2}{2\sigma_k^2}\right)$$
Probability Density Function

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