Events: elementary and compound, E
Sample space: space of all possible events, S
MECE: Mutually exclusive and collectively exhausting events

Probability Axioms:

\[ P(A) \geq 0; \]
\[ P(S) = 1; \]
If \( (E_1 \cap E_2) = 0 \), i.e., if \( E_1 \) and \( E_2 \) exclusive, then \( P(E_1 \cup E_2) = P(E_1) + P(E_2) \)

Probability ~ Frequency \[ P(E) = \lim_{n \to \infty} \frac{\# E = \text{yes}}{\text{total } n} \]

If \( E_2 \subseteq E_1 \), then \( P(E_1) \geq P(E_2) \)

\[ P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2) \]

Recall threat score: \( TS = \frac{P(F = \text{yes} \cap Ob = \text{yes})}{P(F = \text{yes} \cup Ob = \text{yes})} \)

Conditional Probability: “probability of \( E_1 \) given that \( E_2 \) has happened”

\[ P(E_1 | E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)} \]

Independent events: \( P(E_1 \cap E_2) = P(E_1)P(E_2) \)

This means that \( P(E_1 | E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)} = P(E_1) \)

i.e., the probability of \( E_1 \) happening is independent of whether \( E_2 \) happened (e.g., the probability of a summer storm is independent from the phases of the moon).
Exercise: From the Penn State station data for January 1980, compute the probability of precipitation, of T>32F, conditional probability of pp if T>32F, and conditional probability of pp tomorrow if it is raining today.

Exercise: Prove graphically the DeMorgan Laws:

\[ P\{(A \cup B)^C\} = P\{A^C \cap B^C\}; P\{(A \cap B)^C\} = P\{A^C \cup B^C\} \]

Total probability:

\[ P(A) = \sum_{i=1}^{I} P(A \cap E_i) = \sum_{i=1}^{I} P(A | E_i)P(E_i) \] where \( E_i \) are MECE.

Bayes Theorem: It “inverts” the probability

\[ P(E_i | A) = \frac{P(E_i \cap A)}{P(A)} = \frac{P(A | E_i)P(E_i)}{P(A)} = \frac{P(A | E_i)P(E_i)}{\sum_{j=1}^{I} P(A | E_j)P(E_j)} \]

Combines prior information with new information

Example of Bayesian reasoning:
Relationship between pp over SE US and El Niño

Precip. Events: \( E_1 \) (above), \( E_2 \) (normal), \( E_3 \) (below) are MECE. \( A \) is El Niño

Prior information (from past statistics):

\[ P(E_1) = P(E_2) = P(E_3) = 33\% \]
\[ P(A | E_1) = 40\%; P(A | E_2) = 20\%; P(A | E_3) = 0\%; \]

Total probability of \( A \):

\[ P(A) = P(A | E_1)P(E_1) + P(A | E_2)P(E_2) + P(A | E_3)P(E_3) = \]
\[ P(A) = 40\% 33\% + 20\% 33\% + 0\% 33\% = 20\% \]

Bayes, new information: El Niño is happening!!
What is the probability of above normal precipitation?
Note the clear interpretation from the figure: once you
know $A$ is true, the prob. of $E_1$ is 2/3.

$$P(E_1 \mid A) = \frac{P(A \mid E_1)}{P(A)} = \frac{40\%}{20\%} = 66\%!$$

Example of Bayesian use in variational data assimilation:

Prior knowledge (measurement or forecast) $T_1$ of the true value $T$

New measurement: $T_2$.

$$P(T \mid T_2) = \frac{P(T_2 \mid T)P_{\text{prior, given } T_1}(T)}{P(T_2)} = \frac{1}{\sqrt{2\pi \sigma_1}} \cdot \frac{1}{\sqrt{2\pi \sigma_2}} \cdot \frac{1}{e^{\frac{(T_2 - T)^2}{2\sigma_2^2}}} \cdot \frac{1}{e^{\frac{(T - T_1)^2}{2\sigma_1^2}}}$$

Note that the total probability of a measurement $T_2$ given a
climatological average $\overline{T}$ is independent of $T$.

We choose as our best estimate of the true temperature $T$ the value
that maximizes (over $T$) the probability $P(T \mid T_2)$. Since the logarithm is
monotonic, it is equivalent to maximizing (over $T$) the
$$\log P(T \mid T_2) = \text{const} - \frac{(T_2 - T)^2}{2\sigma_2^2} - \frac{(T - T_1)^2}{2\sigma_1^2}$$

or minimize (over $T$) the cost function used in 3D-Var:

$$J = \frac{(T - T_2)^2}{2\sigma_2^2} + \frac{(T - T_1)^2}{2\sigma_1^2}.$$