To solve the linear optimization problems, we can take two approaches:

1. Matrix methods: knowing the solution form that requires inversion, find an efficient method for solving the system.
2. Iterative methods: instead of actually solving for the solution, optimize the cost function.

To solve the nonlinear optimization problem (i.e., nonlinear observation operators), we can either
   - Linearize
   - Solve as a nonlinear optimization problem using the iterative method.

See Exercise 1.
a. Matrix Methods

- Approach: LSE yields to solving
  \[ \mathbf{Ax} = \mathbf{b} \]

  where \( \mathbf{A} = (\alpha \mathbf{I} + \mathbf{HH}^T) \) and \( \mathbf{b} = \mathbf{H}^T \mathbf{y} \)

- Algorithms: many / computational software available for matrix methods to solve the linear systems, for example
  - Cholesky decomposition: Adaptation of LU decomposition
    \[
    \mathbf{A} = \mathbf{LL}^T \rightarrow \mathbf{Ax} = \mathbf{b} \rightarrow \mathbf{LL}^T \mathbf{x} = \mathbf{b} \rightarrow \mathbf{Ly} = \mathbf{b} \quad \& \quad \mathbf{L}^T \mathbf{x} = \mathbf{y}
    \]
  - Orthogonal decomposition
    - QR decompositiion: \( \mathbf{R}, \mathbf{Q} \): upper triangle & orthogonal matrices
      \[
      \mathbf{A} = \mathbf{QR} \rightarrow \mathbf{QRx} = \mathbf{b} \rightarrow \mathbf{Rx} = \mathbf{Q}^T \mathbf{b}
      \]
    - Singular Value Decomposition with diagonal \( \mathbf{S} \)
      \[
      \mathbf{A} = \mathbf{USU}^T \rightarrow \mathbf{USU}^T \mathbf{x} = \mathbf{b} \rightarrow \mathbf{x} = \mathbf{US}^{-1} \mathbf{U}^T \mathbf{b}
      \]

solve directly.
b. Iterative Methods

- **Approach:** Instead of solving for the linear system, the cost function can be iteratively minimized.

- **Conditions:** Any iterative method that solves for the minimum must satisfy the following two conditions: $J(x)$
  - Monotonically decreasing sequence: $J(x^{(k+1)}) < J(x^{(k)})$
    - So that $\Delta x^{(k)} = x^{(k+1)} - x^{(k)}$ is a descent direction.
  - Convergence to: $\lim_{k \to \infty} x^{(k)} = x^{\text{LSE}}$
    - Which satisfies the minimum condition:
      - $\lim_{k \to \infty} \nabla f(x^{(k)}) = \nabla f(x^{\text{LSE}}) = 0$
      - $\lim_{k \to \infty} \nabla^2 f(x^{(k)}) = \nabla^2 f(x^{\text{LSE}}) = \text{positive def. } \in \mathcal{R}^{N \times N}$

- **Algorithms:** many techniques/software available to iterative solve the linear systems, for example
  - Steepest descent methods
  - Conjugate direction/gradient method
  - Newton/quasi-Newton method

solve iteratively
Exercise 1. Implementation of Optimization Methods

Expected for Upcoming Projects:  

No Due Date

1. Implement at least one each for matrix and iterative methods
   - Matrix methods
     • Choleskly decomposition: Adaptation of LU decomposition
     • Orthogonal decomposition
       - QR decomposition
       - Singular Value Decomposition
   - Iterative methods
     • Steepest Descent Method
     • Conjugate direction/gradient method
     • Newton’s/quasi-Newton’s method

2. Validate your codes against the known problem/solution sets.