Project II. Preconditioning for Optimization Problem

- Issue: Minimization may be computationally very inefficient (fig. a)

- Idea: Linear coordinate change $x \rightarrow x^b_k + \Delta x \rightarrow x^b_k + V w$ ($\Delta x = Vw$) to lower the aspect ratio (fig. b)

  - Using SVD of $P^b = US^2U^T$ where $(U, S) \sim$ (rotation, aspect ratio)
    
    \[
    (x-x^b_k)^T(P^b)^{-1}(x-x^b_k) = \Delta x^T (P^b)^{-1} \Delta x = \Delta x^T US^{-2} U^T \Delta x = (S^{-1} U^T \Delta x)^T (S^{-1} U^T \Delta x)
    \]

  - Choose $V^{-1} \Delta x = w$ (or $\Delta x = Vw$) with $V^{-1} = S^{-1} U^T$ (or $V = US$) so that
    
    \[
    (x-x^b_k)^T(P^b)^{-1}(x-x^b_k) = w^T w
    \]

a) Minimization can require many iterations, due to the high aspect ratio.

b) Minimization requires much less iterations when the aspect ratio is about 1
Preconditioned optimization problem

- Cost function using the control \( \mathbf{w} \)
  \[
  J(\mathbf{w}) = J^b(\mathbf{w}) + J^o(\mathbf{w})
  \]
  \[
  J^b(\mathbf{w}) = \frac{1}{2} \mathbf{w}^T \mathbf{w}
  \]
  \[
  J^o(\mathbf{w}) = \frac{1}{2}(\mathbf{d}_k^o - \mathbf{HVw})^T(\mathbf{R}^o)^{-1}(\mathbf{d}_k^o - \mathbf{HVw})
  \]
  \[
  \mathbf{d}_k^b = \mathbf{y}_k^o - \mathbf{Hx}_k^b
  \]

- Analysis in \( \mathbf{x} \)
  \[
  \mathbf{x}_k^a = \mathbf{x}_k^b + \mathbf{Vw}_k^a
  \]

- Note
  - Aspect ratio of \( \mathbf{P}^b \)-ellipse is given by the ratio of the diagonal element of \( \mathbf{S} \).
    Ratio of the largest singular value to the smallest relates to:
    \[
    \text{condition number of } \mathbf{P}^b = \left( \frac{S_{11}}{S_{NN}} \right)^2 \quad \text{for } S_{11} > \cdots > S_{NN} \text{ with } S_{nn} > 0
    \]
  - Preconditioning process
    - Let you confirm that \( \mathbf{P}^b \) is positive definite, i.e., \( S_{nn} > 0 \) are all real.
    - May not be a “perfect” solution to the optimization, but it helps.