Exercise 3. Preconditioning for Optimization Problem

Expected for Upcoming Projects: No Due Date

- Issue: Minimization may be computationally very inefficient (fig. a)

a) Minimization can require many iterations, due to the high aspect ratio.

b) Minimization requires much less iterations when the aspect ratio is $\sim 1$

- Idea: Linear coordinate change $x \rightarrow x^b + \Delta x \rightarrow x^b + Vw$ ($\Delta x = Vw$)
  to lower the aspect ratio (fig. b)

  - Using SVD of $P^b = USU^T$ where $(U, S) \sim$ (rotation, aspect ratio)
    $(x-x^b)^T(P^b)^{-1}(x-x^b) = \Delta x^T(P^b)^{-1}\Delta x = \Delta x^T US^{-2}U^T \Delta x = (S^{-1}U^T \Delta x)^T(S^{-1}U^T \Delta x)$

  - Choose $V^{-1}\Delta x = w$ (or $\Delta x = Vw$) with $V^{-1} = S^{-1}U^T$ (or $V = US$) so that
    $(x-x^b)^T(P^b)^{-1}(x-x^b) = w^T w$

- Exercise: Development of the optimization scheme in 2 ways
  1. Original: Optimize $J(x)$
  2. Preconditioning: Optimize $J(w)$ and then transform $x^b + \Delta x \rightarrow x^b + Vw$