Exercise 4. Tangent Linear Model

Expected for Upcoming Projects: No Due Date

- Build the Tangent Linear Model (TLM) of the nonlinear model
  \[ \frac{d}{dt} x = f(x, t) \]
  along a reference trajectory \( x^{c}(t; x^{c}_{0}, t_{0}) \) where
  \[ \frac{d}{dt} x^{c} = f(x^{c}, t) \]
  Tangent linear model is given by
  - Time continuous: \( F^{c}(t; x^{c}_{0}, t_{0}) \) for any \( t \)
    \[ F^{c} = F^{c}(t; x^{c}_{0}, t_{0}) = \frac{d}{dx} f(x, t) \bigg|_{(x^{c}, t)} \in \mathbb{R}^{NW} \]
  - Time discrete: \( M^{c}_{k,k-1}(x^{c}_{0}, t_{0}) \) & \( M^{c}_{k,0}(x^{c}_{0}, t_{0}) \) for \( k = 1, \ldots, K \)
    \[ \frac{d}{dt} M^{c}(t; t_{k-1}) = F^{c}(t; t_{k-1}) \Rightarrow M^{c}(t; t_{k-1}) = 1 + \int_{t_{k-1}}^{t} F^{c}(\tau; t_{k-1}) d\tau \]
    \[ M^{c}_{k,k-1} = M^{c}(t_{k}; t_{k-1}) \]
    \[ M^{c}_{k,0} = M^{c}(t_{1}; t_{0}) \]
    \[ M^{c}_{k,0} = \prod_{i=1}^{K} M^{c}_{k,i} \]
    \[ M^{c}_{k,0} = \prod_{i=1}^{K} M^{c}_{k,i} \]

- Validate TLM
  - Instantaneously at \( t_{0} \):
    Check properties of \( F^{c} \), based on the model & against analytical values
      - Lorenz systems: periodicity etc
      - PV system: action-interaction etc
  - For \( k = 0, \ldots, K \) (in time) with \( \Delta x_{k} = \delta x_{0} \)
    Compare \( \Delta x_{k} \) by TLM with nonlinear error growth with \( \delta x_{k} \)
    \[ \delta x = m_{k} \left( x^{c} + \delta x^{c} \right) - m_{k} x^{c} \]
    - Error growth: Individual & difference: \( | \Delta x_{k} |, | \delta x_{k} |, | \Delta x_{k} - \delta x_{k} | \)
    - Growth rate: \( | \Delta x_{k} | / | \delta x_{k} | = 1 \) [or comparison for single variable]
    - Growth direction: \( | \Delta x_{k} | / | \delta x_{k} | = 1 \) [cos-angle]
    [Note]
    - Increase \( k \), and study how \( \Delta x_{k} \) and \( \delta x_{k} \) separate
    - Try different \( \Delta x_{0} \)
    - Try different \( x^{c} \)